

Some new estimates for the moments of guessing mappings

S. S. DRAGOMIR*

Abstract. *In this paper, by the use of some new analytic inequalities for arithmetic means, we point out new estimation for the moments of guessing mapping which complement in a natural way the recent results of Arikan [2], Boztas [3] and Dragomir, Van der Hoek [4]-[6].*

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1. Introduction

J.L. Massey in the paper [1] considered the problem of guessing the value of realization of a random variable X by asking questions of the form: "Is X equal to x ?" until the answer is "Yes".

Let $G(X)$ denote the number of guesses required by a particular guessing strategy when $X = x$.

Massey observed that $E(G(X))$, the average number of guesses is minimized by a guessing strategy that guesses the possible values of X in decreasing order of probability.

We begin by giving a formal and generalized statement of the above problem by following E. Arikan [2].

Let (X, Y) be a pair of random variable with X taking values in a finite set \mathcal{X} of size n , Y taking values in a countable set \mathcal{Y} . Call a function $G(X)$ of the random variable X a *guessing function* in X if $G : \mathcal{X} \rightarrow \{1, \dots, n\}$ is one-to-one. Call a function $G(X | Y)$ a *guessing function for X given Y* if for any fixed value $Y = y$, $G(X | y)$ is a guessing function for X . $G(X | y)$ will be thought of as the number of guessing required to determine X when the value of Y is given.

The following inequalities on the moments of $G(X)$ and $G(X | Y)$ were proved by E. Arikan in the recent paper [2].

*School of Communications and Informatics, Victoria University of Technology, PO Box, 14428 MC Melbourne, 8001 Victoria, Australia, e-mail: sever@matilda.vu.edu.au

Theorem 1. For an arbitrary guessing function $G(X)$ and $G(X|Y)$ and any $p \geq 0$, we have:

$$E(G(X)^p) \geq (1 + \ln n)^{-p} \left[\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+p}} \right]^{1+p} \quad (1)$$

and

$$E(G(X|Y)^p) \geq (1 + \ln n)^{-p} \sum_{y \in \mathcal{Y}} \left[\sum_{x \in \mathcal{X}} P_{X,Y}(x, y)^{\frac{1}{1+p}} \right]^{1+p} \quad (2)$$

where $P_{X,Y}$ and P_X are probability distributions of (X, Y) and X , respectively.

In paper [6], S.S. Dragomir and J. van der Hoek have proved the following result for the moments of guessing mapping:

Theorem 2. Assume that $P_M := \max \{p_i \mid i = 1, \dots, n\}$ and $P_m := \min \{p_i \mid i = 1, \dots, n\}$ and $P_M \neq P_m$. Then we have the estimates:

$$\begin{aligned} & G_p(n) \left[P_m n^{p+1} + \frac{1}{(P_M - P_m)^p} (1 - n P_m)^{p+1} \right] \\ & \leq E(G(X)^p) \leq G_p(n) \left[P_M n^{p+1} + \frac{1}{(P_M - P_m)^p} (n P_M - 1)^{p+1} \right] \end{aligned}$$

for $p \geq 1$, where $G_p(n) := \frac{S_p(n)}{n^{p+1}}$ and $S_p(n) := \sum_{i=1}^n i^p$.

Corollary 1. With the above assumption, we have:

$$\begin{aligned} \frac{1}{2} \left(1 + \frac{1}{n} \right) \frac{P_m P_M n^2 - 2n P_m + 1}{P_M - P_m} & \leq E(G(X)) \\ & \leq \frac{1}{2} \left(1 + \frac{1}{n} \right) \frac{-P_m P_M n^2 + 2P_M n - 1}{P_M - P_m}. \end{aligned}$$

For other estimations of $E(G(X)^p)$ see the papers [4] - [6].

The main aim of this paper is to point out different estimations of the moments $E(G(X)^p)$ by the use of some new inequalities for arithmetic means which will be pointed out in the next section.

2. Some analytic inequalities

We shall start with the following lemma of "summation by parts" which is well known in the literature (see for example [7, p. 26]):

Lemma 1. Let $a_i, b_i \in \mathbf{R}$ ($i = 1, \dots, n$) and denote $\Delta a_i := a_{i+1} - a_i$ ($i = 1, \dots, n$). Then we have the inequality:

$$\sum_{i=1}^{n-1} b_i \Delta a_i = a_n b_n - a_1 b_1 - \sum_{i=1}^{n-1} a_{i+1} \Delta b_i. \quad (3)$$

The following corollary holds:

Corollary 2. *If $t_i, z_i \in \mathbf{R}$ ($i = 1, \dots, n$) and $T_k = \sum_{i=1}^k t_i$, $T_0 := 0$ ($k = 1, \dots, n$), then we have the identity:*

$$\sum_{i=1}^n t_i z_i = T_n z_n - \sum_{i=1}^n T_i \Delta z_i. \quad (4)$$

Now, let us consider the arithmetic means:

$$A_n(p, x) := \sum_{i=1}^n t_i x_i \text{ where } p_i \geq 0 \text{ and } \sum_{i=1}^n p_i = 1,$$

$$A_n(q, x) := \sum_{i=1}^n q_i x_i \text{ where } q_i \geq 0 \text{ and } \sum_{i=1}^n q_i = 1$$

and $x = (x_i)_{i=1, \dots, n}$, $x_i \in \mathbf{R}$ ($i = 1, \dots, n$).

We are interested here to establish some estimations for the difference $A_n(p, x) - A_n(q, x)$ in terms of p, q and x .

Theorem 3. *With the above assumptions for the sequences p_i, q_i, x_i ($i = 1, \dots, n$) we have the inequality:*

$$\begin{aligned} & |A_n(p, x) - A_n(q, x)| \\ & \leq \begin{cases} \max_{i=1, \dots, n-1} |x_{i+1} - x_i| \sum_{i=1}^{n-1} |P_i - Q_i|, \\ \left(\sum_{i=1}^{n-1} |P_i - Q_i|^s \right)^{\frac{1}{s}} \left(\sum_{i=1}^{n-1} |x_{i+1} - x_i|^l \right)^{\frac{1}{l}}, \frac{1}{s} + \frac{1}{l} = 1, 1 < s < \infty, \\ \max_{i=1, \dots, n-1} |P_i - Q_i| \sum_{i=1}^{n-1} |x_{i+1} - x_i|, \end{cases} \quad (5) \end{aligned}$$

where

$$P_i := \sum_{k=1}^i p_k \text{ and } Q_i := \sum_{k=1}^i q_k \text{ for } i \in \{1, \dots, n\}.$$

Proof. Using the identity (4) we have :

$$\begin{aligned} |A_n(p, x) - A_n(q, x)| &= \left| \sum_{i=1}^n (p_i - q_i) x_i \right| = \left| x_n \sum_{i=1}^n (p_i - q_i) - \sum_{i=1}^{n-1} (P_i - Q_i) \Delta x_i \right| \\ &= \left| \sum_{i=1}^{n-1} (P_i - Q_i) \Delta x_i \right| \leq \sum_{i=1}^{n-1} |P_i - Q_i| |x_{i+1} - x_i|. \end{aligned}$$

Now, let us remark that the first and the last inequality in (5) are obvious.

The second inequality follows by the discrete Hölder's inequality.

We shall omit the details. \square

Corollary 3. *With the above assumptions we have that*

$$|A_n(p, x) - A_n(x)| \leq \begin{cases} \frac{\max_{i=1, n-1} |x_{i+1} - x_i|}{n} \sum_{i=1}^{n-1} |P_i - \frac{i}{n}|, \\ \left(\sum_{i=1}^{n-1} |P_i - \frac{i}{n}|^s \right)^{\frac{1}{s}} \left(\sum_{i=1}^n |x_{i+1} - x_i|^l \right)^{\frac{1}{l}}, \frac{1}{s} + \frac{1}{l} = 1, 1 < s < \infty, \\ \frac{\max_{i=1, n-1} |P_i - \frac{i}{n}|}{n} \sum_{i=1}^{n-1} |x_{i+1} - x_i|, \end{cases} \quad (6)$$

where by $A_n(x)$ we denote the unweighted arithmetic mean, i.e.,

$$A_n(x) = \frac{1}{n} \sum_{i=1}^n x_i.$$

Another type of estimation can be found in the following theorem too.

Theorem 4. *Under the assumption of Theorem 3, we have*

$$|A_n(p, x) - A_n(x) - X_n(p_n - q_n)| \leq \begin{cases} \frac{\max_{i=1, n-1} |\Delta(p_i - q_i)|}{n} \sum_{i=1}^{n-1} |X_i|; \\ \left(\sum_{i=1}^{n-1} |\Delta(p_i - q_i)|^s \right)^{\frac{1}{s}} \left(\sum_{i=1}^{n-1} |X_i|^l \right)^{\frac{1}{l}}, \frac{1}{s} + \frac{1}{l} = 1, 1 < s < \infty; \\ \frac{\max_{i=1, n-1} |X_i|}{n} \sum_{i=1}^{n-1} |\Delta(p_i - q_i)|; \end{cases} \quad (7)$$

where X_i is given by

$$X_i := \sum_{k=1}^i x_k, i \in \{1, \dots, n\}.$$

Proof. Using the identity (4) we can write :

$$|A_n(p, x) - A_n(q, x)| = \sum_{i=1}^n (p_i - q_i) x_i = X_n(p_n - q_n) - \sum_{i=1}^{n-1} X_i \Delta(p_i - q_i)$$

from where we get

$$A_n(p, x) - A_n(q, x) - X_n(p_n - q_n) = - \sum_{i=1}^{n-1} X_i \Delta(p_i - q_i)$$

and then we have the estimation :

$$|A_n(p, x) - A_n(q, x) - X_n(p_n - q_n)| \leq \sum_{i=1}^{n-1} |X_i| |\Delta(p_i - q_i)|$$

which, as above, imply the desired inequality (7). \square

$$\left\{ \begin{array}{l} \max_{i=1, n-1} |\Delta p_i| \sum_{i=1}^{n-1} |X_i|, \\ \left(\sum_{i=1}^{n-1} |\Delta p_i|^s \right)^{\frac{1}{s}} \left(\sum_{i=1}^{n-1} |X_i|^t \right)^{\frac{1}{t}}, \frac{1}{s} + \frac{1}{t} = 1, 1 < s < \infty, \\ \max_{i=1, n-1} |X_i| \sum_{i=1}^{n-1} |\Delta p_i|. \end{array} \right. \quad (8)$$

Lemma 2. *Let $a_i, b_i \in \mathbf{R}$ ($i = 1, \dots, n$) be so that*

$$a \leq a_i \leq A, b \leq b_i \leq B$$

for all $i = 1, \dots, n$. Then we have the inequality:

$$\left| \frac{1}{n} \sum_{i=1}^n a_i b_i - \frac{1}{n} \sum_{i=1}^n a_i \cdot \frac{1}{n} \sum_{i=1}^n b_i \right| \leq \frac{1}{4} (A - a)(B - b). \quad (9)$$

If $h, g : [a, b] \rightarrow \mathbf{R}$ are two integrable functions so that $m_1 \leq g(x) \leq M_1$, $m_2 \leq h(x) \leq M_2$ for all $x \in [a, b]$, then we have the estimation:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b g(x) h(x) dx - \frac{1}{b-a} \int_a^b g(x) dx \cdot \frac{1}{b-a} \int_a^b h(x) dx \right| \\ & \leq \frac{1}{4} (M_1 - m_1) (M_2 - m_2). \end{aligned} \quad (10)$$

Now, if we are choosing in (10)

$$g(x) = \begin{cases} a_1, x \in [0, 1) \\ a_2, x \in [1, 2) \\ \\ a_n, x \in [n-1, n] \end{cases}$$

and

[illegible]

then $m_1 = a, M_1 = A, m_2 = b$ and $M_2 = B$ and obviously

$$\int_0^n g(x) dx = \sum_{i=1}^n a_i, \int_0^n h(x) dx = \sum_{i=1}^n b_i$$

and

$$\int_0^n g(x) h(x) dx = \sum_{i=1}^n a_i b_i$$

and the lemma is proved. \square

We are able now to point out some different estimations for the modulus of the difference $A_n(p, x) - A_n(q, x)$.

Theorem 5. *With the above assumptions for p, q and X , we have the inequality:*

$$\left| A_n(p, x) - A_n(q, x) - \frac{1}{n-1} (x_n - x_1) \sum_{i=1}^{n-1} (n-i) (p_i - q_i) \right| \quad (11)$$

$$\leq \frac{n-1}{4} (\Gamma - \gamma) (\Delta - \delta)$$

provided that:

$$\delta \leq x_i \leq \Delta \text{ for all } i = 1, \dots, n-1$$

and

$$\gamma \leq P_i - Q_i \leq \Gamma \text{ for all } i = 1, \dots, n-1.$$

Proof. Choose in Lemma 2

$$a_i := P_i - Q_i, b_i = \Delta x_i, i = 1, \dots, n-1.$$

Then we have:

$$\left| \frac{1}{n-1} \sum_{i=1}^{n-1} (P_i - Q_i) \Delta x_i - \frac{1}{n-1} \sum_{i=1}^{n-1} (P_i - Q_i) \cdot \frac{1}{n-1} \sum_{i=1}^{n-1} \Delta x_i \right|$$

$$\leq \frac{1}{4} (\Delta - \delta) (\Gamma - \gamma).$$

As we have:

$$\sum_{i=1}^{n-1} (P_i - Q_i) = \sum_{i=1}^{n-1} (n-i) (p_i - q_i),$$

$$\sum_{i=1}^{n-1} \Delta x_i = x_n - x_1$$

and

$$\sum_{i=1}^{n-1} (p_i - q_i) x_i = \sum_{i=1}^{n-1} (P_i - Q_i) \Delta x_i$$

then (see *Theorem 3*), we get that

$$\begin{aligned} & \left| A_n(p, x) - A_n(q, x) - \frac{1}{n-1} (x_n - x_1) \sum_{i=1}^{n-1} (n-i) (p_i - q_i) \right| \\ &= \left| \sum_{i=1}^{n-1} (P_i - Q_i) \Delta x_i - \frac{1}{n-1} \sum_{i=1}^{n-1} (P_i - Q_i) \cdot \frac{1}{n-1} \sum_{i=1}^{n-1} \Delta x_i \right| \\ &\leq \frac{n-1}{4} (\Gamma - \gamma) (\Delta - \delta) \end{aligned}$$

and the estimation is proved. \square

Corollary 5. *With the above assumptions for X and if*

$$\tilde{\gamma} \leq P_i - \frac{i}{n} \leq \tilde{\Gamma}$$

then

$$\begin{aligned} & \left| A_n(p, x) - A_n(q, x) - \frac{1}{n-1} (x_n - x_1) \sum_{i=1}^{n-1} (n-i) \left(p_i - \frac{1}{n}\right) \right| \\ &\leq \frac{n-1}{4} (\tilde{\Gamma} - \tilde{\gamma}) (\Delta - \delta). \end{aligned} \tag{12}$$

The following theorem also holds.

Theorem 6. *Suppose that p, q, x satisfy the condition:*

$$x \leq X_i \leq X, i = 1, \dots, n-1$$

and

$$\varphi \leq \Delta(p_i - q_i) \leq \Phi, i = 1, \dots, n-1.$$

Then we have:

$$\begin{aligned} & \left| A_n(p, x) - A_n(q, x) - X_n(p_n - q_n) + \frac{1}{n-1} [p_n - p_1 - (q_n - q_1)] \sum_{i=1}^{n-1} (n-i) x_i \right| \\ &\leq \frac{n-1}{4} (X - x) (\Phi - \varphi). \end{aligned} \tag{13}$$

Proof. Let us apply the discrete Grüss' inequality (2.7) for $a_i := X_i$, $b_i := \Delta(p_i - q_i)$, $i = 1, \dots, n-1$ to get:

$$\left| \frac{1}{n-1} \sum_{i=1}^{n-1} X_i \Delta(p_i - q_i) - \frac{1}{n-1} \sum_{i=1}^{n-1} X_i \cdot \frac{1}{n-1} \sum_{i=1}^{n-1} \Delta(p_i - q_i) \right|$$

$$\leq \frac{1}{4} (X - x) (\Phi - \varphi).$$

But

$$\sum_{i=1}^{n-1} X_i = \sum_{i=1}^{n-1} (n-i) x_i,$$

$$\sum_{i=1}^{n-1} \Delta(p_i - q_i) = p_n - q_n - p_1 + q_1 = p_n - p_1 - (q_n - q_1),$$

and then we get:

$$\left| - \sum_{i=1}^{n-1} X_i \Delta(p_i - q_i) + \frac{1}{n-1} (p_n - p_1 - (q_n - q_1)) \sum_{i=1}^{n-1} (n-i) x_i \right| \quad (14)$$

$$\leq \frac{n-1}{4} (X - x) (\Phi - \varphi).$$

But,

$$- \sum_{i=1}^{n-1} X_i \Delta(p_i - q_i) = A_n(p, x) - A_n(q, x) - X_n(p_n - q_n)$$

(see the proof of *Theorem 4*) and then by (14) we obtain:

$$\left| A_n(p, x) - A_n(q, x) - X_n(p_n - q_n) + \frac{1}{n-1} (p_n - p_1 - (q_n - q_1)) \sum_{i=1}^{n-1} (n-i) x_i \right|$$

$$\leq \frac{n-1}{4} (X - x) (\Phi - \varphi)$$

and the theorem is thus proved. \square

The following corollary also holds.

Corollary 6. *If x and p satisfy the condition:*

$$x \leq X_i \leq X, i = 1, \dots, n-1$$

and

$$\tilde{\varphi} \leq \Delta p_i \leq \tilde{\Phi}, i = 1, \dots, n-1$$

then

$$\begin{aligned} & \left| A_n(p, X) - A_n(X) - X_n \left(p_n - \frac{1}{n} \right) + \frac{1}{n-1} (p_n - p_1) \sum_{i=1}^{n-1} (n-i) x_i \right| \\ & \leq \frac{n-1}{4} (X - x) (\tilde{\Phi} - \tilde{\varphi}). \end{aligned} \quad (15)$$

3. Applications to the moments of guessing mapping

To simplify the notation further, we assume that the x_i are numbered such that x_k is always the k^{th} guess with respect to G . This yields

$$E(G^p) = \sum_{i=1}^n i^p p_i, (p \geq 0).$$

Now, if we consider another guessing mapping L , we can write

$$E(L^p) = \sum_{i=1}^n i^p p_{\sigma(i)}$$

where σ is a permutation of the indices $\{1, \dots, n\}$ associated with L .

Using the results from Section 2 we can give the following theorems.

Theorem 7. *Let $G(X)$ and $L(X)$ be two guessing mappings associated with the random variable X and $E(G(X)^p)$, $E(L(X)^p)$ ($p \geq 1$) their p -moments. Then we have the estimation:*

$$\begin{aligned} & |E(G(X)^p) - E(L(X)^p)| \\ & \leq \begin{cases} [n^p - (n-1)^p] \sum_{i=1}^{n-1} |P_i - P_{\sigma(i)}|, \\ \left(\sum_{i=1}^{n-1} |P_i - P_{\sigma(i)}|^s \right)^{\frac{1}{s}} \left(\sum_{i=1}^{n-1} [(i+1)^p - i^p]^t \right)^{\frac{1}{t}} \frac{1}{s} + \frac{1}{t} = 1, 1 < s < \infty, \\ \max_{i=1, n-1} |P_i - P_{\sigma(i)}| (n^p - 1), \end{cases} \end{aligned} \quad (16)$$

where $P_i = \sum_{k=1}^i p_k$ and $P_{\sigma(i)} = \sum_{k=1}^i p_{\sigma(k)}$.

Proof. Let us choose in *Theorem 3*, $q_i = p_{\sigma(i)}$ and $x_i = i^p$. As for $p \geq 1$ the sequence x_i is convex, we have that $(x_{i+1} - x_i)_{i=1, n-1}$ is increasing, and then

$$\max_{i=1, n-1} |x_{i+1} - x_i| = n^p - (n-1)^p.$$

Also, as $x_{i+1} \geq x_i$, we have $\sum_{i=1}^{n-1} |x_{i+1} - x_i| = n^p - 1$.

Now, by the inequality (5) we get (16). \square

Remark 1. If we choose $p = 1$, $s = l = 2$, then we have the estimation:

$$|E(G(X)) - E(L(X))| \leq \begin{cases} \sum_{i=1}^{n-1} |P_i - P_{\sigma(i)}|, \\ (n-1)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} |P_i - P_{\sigma(i)}|^2 \right)^{\frac{1}{2}}, \\ (n-1) \max_{i=1, n-1} |P_i - P_{\sigma(i)}|. \end{cases} \quad (17)$$

Remark 2. If we choose $p = 2$, $s = l = 2$, then we have the estimation:

$$|E(G(X)^2) - E(L(X)^2)| \leq \begin{cases} (2n-1) \sum_{i=1}^{n-1} |P_i - P_{\sigma(i)}|, \\ \left[\frac{(n-1)(4n^2+4n+3)}{3} \right]^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} |P_i - P_{\sigma(i)}|^2 \right)^{\frac{1}{2}}, \\ (n^2-1) \max_{i=1, n-1} |P_i - P_{\sigma(i)}|. \end{cases} \quad (18)$$

Indeed, as a simple calculation shows us that

$$\begin{aligned} \sum_{i=1}^{n-1} [(i+1)^2 - i^2]^2 &= \sum_{i=1}^{n-1} (2i+1)^2 = 4 \sum_{i=1}^{n-1} i^2 + 4 \sum_{i=1}^{n-1} i + n-1 \\ &= 4 \frac{(n-1)n(2n-1)}{6} + \frac{4(n-1)n}{2} + n-1 \\ &= (n-1) \left[\frac{2n(2n-1)}{3} + 2n+1 \right] \\ &= \frac{(n-1)(4n^2-2n+6n+3)}{3} = \frac{(n-1)(4n^2+4n+3)}{3}. \end{aligned}$$

Another result is embodied in the following theorem.

Theorem 8. With the assumptions of *Theorem 7*, we have the estimation:

$$|E(G(X)^p) - E(L(X)^p) - S_p(n)(p_n - p_{\sigma(n)})| \quad (19)$$

$$\leq \begin{cases} \max_{i=1, n-1} |\Delta p_i - \Delta p_{\sigma(i)}| \sum_{i=1}^{n-1} S_p(i); \\ \left(\sum_{i=1}^{n-1} |\Delta p_i - \Delta p_{\sigma(i)}|^s \right)^{\frac{1}{s}} \left(\sum_{i=1}^{n-1} S_p^l(i) \right)^{\frac{1}{l}}, \frac{1}{s} + \frac{1}{l} = 1, 1 < s < \infty; \\ S_p(n-1) \sum_{i=1}^{n-1} |\Delta p_i - \Delta p_{\sigma(i)}|; \end{cases}$$

where $S_p(k) = \sum_{i=1}^k i^p, k = 1, \dots, n$.

The proof is obvious by *Theorem 4* choosing $x_i = i^p, q_i = p_{\sigma(i)}, i = 1, \dots, n$. We shall omit the details.

Remark 3. If we choose in (19) $p = 1$, we get :

$$\left| E(G(X)) - E(L(X)) - \frac{n(n+1)}{2} (p_n - p_{\sigma(n)}) \right| \quad (20)$$

$$\leq \begin{cases} \frac{(n-1)n(n+1)}{6} \max_{i=1, n-1} |\Delta p_i - \Delta p_{\sigma(i)}| \\ \frac{(n-1)n}{2} \sum_{i=1}^{n-1} |\Delta p_i - \Delta p_{\sigma(i)}| \end{cases}.$$

Indeed, as a simple calculation shows us that:

$$\begin{aligned} \sum_{i=1}^{n-1} S_1(i) &= \sum_{i=1}^{n-1} \frac{i(i+1)}{2} = \frac{1}{2} \left(\sum_{i=1}^{n-1} i^2 + \sum_{i=1}^{n-1} i \right) \\ &= \frac{1}{2} \left[\frac{(n-1)n(2n+1)}{6} + \frac{(n-1)n}{2} \right] = \frac{n(n-1)}{4} \left(\frac{2n-1}{3} + 1 \right) \\ &= \frac{2n(n-1)(n+1)}{12} = \frac{(n-1)n(n+1)}{6}. \end{aligned}$$

A similar inequality can be pointed out if we choose $p = l = 2$ in the second inequality in (19). We shall omit the details.

Now, if we use *Theorem 5*, we can state the following theorem:

Theorem 9. With the assumptions from *Theorem 7*, we have:

$$\begin{aligned} &\left| E(G(X)^p) - E(L(X)^p) - \frac{n^p-1}{n-1} \sum_{i=1}^{n-1} (n-i) (p_i - p_{\sigma(i)}) \right| \\ &\leq \frac{(n-1)(n^p-1)}{4} (M_{P,\sigma} - m_{p,\sigma}), p \geq 1, \end{aligned} \quad (21)$$

where

$$M_{p,\sigma} := \max_{i=1, n-1} |P_i - P_{\sigma(i)}|$$

and

$$m_{p,\sigma} := \min_{i=1, n-1} |P_i - P_{\sigma(i)}|.$$

Finally, by the use of *Theorem 6*, we can state another estimation result.

Theorem 10. *With the above assumptions, we have*

$$\left| E(G(X)^p) - E(L(X)^p) - \frac{nS_p(n) - S_{p+1}(n)}{n-1} [p_n - p_1 - p_{\sigma(n)} + p_{\sigma(1)}] \right| \quad (22)$$

$$\begin{aligned} & -S(n) (p_n - p_{\sigma(n)}) \mid \\ & \leq \frac{(n-1)(S_p(n) - 1)}{4} (\Phi_{p,\sigma} - \varphi_{p,\sigma}) \end{aligned}$$

where

$$\Phi_{p,\sigma} = \max_{i=1, n-1} (\Delta p_i - \Delta p_{\sigma(i)})$$

and

$$\varphi_{p,\sigma} = \min_{i=1, n-1} (\Delta p_i - \Delta p_{\sigma(i)}).$$

Remark 4. *In all previous results we have compared the moments of two guessing mappings $G(X)$ and $L(X)$ and obtain estimations in terms of the corresponding probabilities $p_i(p_{\sigma(i)})$ ($i = 1, \dots, n$).*

In papers [4]-[6], S.S. Dragomir and J. van der Hoek obtained among others some estimations of moments of guessing mapping $E(G(X)^p)$ in terms of $S_p(n) := \sum_{i=1}^{n-1} i^p$ and the probabilities $P_M := \max\{p_i \mid i = 1, \dots, n\}$ and $P_m := \min\{p_i \mid i = 1, \dots, n\}$. Let us recall only one of them:

Theorem 11. *With the above assumptions we have :*

$$\left| E(G(X)^p) - \frac{1}{n} S_p(n) \right| \leq \frac{n(n^p - 1)}{4} (P_M - P_m). \quad (23)$$

Remark 5. *If we put in (23) $p = 1$, we get*

$$\left| E(G(X)) - \frac{n+1}{4} \right| \leq \frac{n(n-1)}{4} (P_M - P_m).$$

If we choose $p = 2$, we get

$$\left| E(G(X)^2) - \frac{(n+1)(2n+1)}{6} \right| \leq \frac{n(n^2 - 1)}{4} (P_M - P_m)$$

and, finally for $p = 3$, we obtain:

$$\left| E \left(G(X)^3 \right) - \frac{n(n+1)^2}{4} \right| \leq \frac{n(n^3-1)}{4} (P_M - P_m).$$

In this way, if we apply *Corollary 3*, we can obtain the following estimation result:

Theorem 12. *Let $G(X)$ be an arbitrary guessing mappig for the random variable X and $p \geq 1$. Then we have the estimation*

$$\begin{aligned} & \left| E \left(G(X)^p \right) - \frac{1}{n} S_p(n) \right| \\ & \leq \begin{cases} [n^p - (n-1)^p] \sum_{i=1}^{n-1} \left| P_i - \frac{i}{n} \right|, \\ \left(\sum_{i=1}^{n-1} \left| P_i - \frac{i}{n} \right|^s \right)^{\frac{1}{s}} \left(\sum_{i=1}^{n-1} [(i+1)^p - i^p]^l \right)^{\frac{1}{l}}, \frac{1}{s} + \frac{1}{l} = 1, 1 < s < \infty, \\ \max_{i=1, n-1} \left| P_i - \frac{i}{n} \right| (n^p - 1). \end{cases} \end{aligned}$$

Remark 6. *If we choose $p = 1$, $s = l = 2$, then we have the estimation*

$$\left| E \left(G(X) \right) - \frac{n+1}{2} \right| \leq \begin{cases} \sum_{i=1}^{n-1} \left| P_i - \frac{i}{n} \right|, \\ n^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \left| P_i - \frac{i}{n} \right|^2 \right)^{\frac{1}{2}}, \\ (n-1) \max_{i=1, n-1} \left| P_i - \frac{i}{n} \right|. \end{cases}$$

Remark 7. *If we choose $p = 2$, $s = l = 2$, then we have*

$$\left| E \left(G(X)^2 \right) - \frac{(n+1)(2n+1)}{6} \right| \leq \begin{cases} (2n-1) \sum_{i=1}^{n-1} \left| P_i - \frac{i}{n} \right|, \\ \left[\frac{(n-1)(4n^2+4n+3)}{3} \right]^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \left| P_i - \frac{i}{n} \right|^2 \right)^{\frac{1}{2}}, \\ (n^2-1) \max_{i=1, n-1} \left| P_i - \frac{i}{n} \right|. \end{cases}$$

Similar results can be obtained, if we are going to apply the other results embodied in *Corollaries 4*, *5* and *6*. We shall omit the details.

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